ALGEBRAIC LOGIC AND LOGICAL GEOMETRY IN ARBITRARY VARIETIES OF ALGEBRAS

BORIS PLOTKIN

The paper consists of two parts. The first part is devoted to logic for universal algebraic geometry. The second one deals with problems and some results. It may be regarded as a brief exposition of some ideas from the book [18].

1. Logic for Universal Algebraic Geometry

1.1. **Getting started.** For me personally, the topic of this paper originates from two main sources. The first one was my interest to mathematical models in knowledge theory, knowledge bases and databases, see, in particular, [13], [12], [15], [16].

Let us describe briefly how the bridge between algebra and knowledge theory works. We consider the following three components of knowledge:

(1) A syntactical part of knowledge, based on a language of the given logic, is *the description of knowledge.*
(2) *The subject of knowledge* is an object in the given applied field, i.e., an object for which we determine knowledge. In algebraic terms the subject of knowledge is presented by an algebra H in a variety Θ or by a model over this algebra.
(3) *The content of knowledge* (its semantics). Using some abuse of language we can consider the content of knowledge as a reply to the query to a knowledge base.

A certain category of formulas in algebraic logic is related to a knowledge description. We consider a knowledge description as a system of equations or, more generally, a system of formulas. It corresponds the knowledge content, which consists of solutions of the given system. These solutions are presented by points in the corresponding affine space. The category of knowledge content having definable sets in the affine space as objects, is defined in a natural way. Any passage from the knowledge description to the knowledge content is determined by a functor from the first category to the second. This functor depends on the subject of knowledge.

All these notions are defined with respect to some variety of algebras Θ. In algebraic setting a variety Θ is a counter-part of the notion of a data type defined for databases.

The described approach to knowledge theory motivates studies in logical geometry.

Another inspiration is related to acquaintance with the works of E.Rips and Z.Sela particularly presented at the Amitsur Seminar in Jerusalem, and also with the works of V.Remeslennikov, O.Kharlampovich, A.Myasnikov and others. In these papers algebraic geometry in free groups has been developed. In parallel, within many years I was influenced by general viewpoints of A.Tarski and A.Maltsev on elementary theories of algebras and models.

Both sources described above gave rise to the idea of universal algebraic geometry (UAG). In UAG we try to transit from classical algebraic geometry associated with the variety of commutative and associative algebras over a field to geometry and logic in an arbitrary variety Θ and fixed algebra $H \in \Theta$. The case when Θ is the variety of groups and H is a free group in Θ remains of principal importance in UAG. We shall note that the general viewpoint from the positions of UAG to the classical variety provides some new tasks in the classical geometry as well.

1.2. **Points and spaces of points.** Let an algebra $H \in \Theta$ be given. Take a finite set of variables $X = \{x_1, \ldots, x_n\}$. Define points as maps of the form $\mu : X \to H$. Each point μ determines the sequence $(h_1, \ldots, h_n)$, where $h_i = \mu(x_i)$. Since we are working in a given variety Θ, one also can define a point μ as a homomorphism

$$\mu : W(X) \to H,$$

where $W(X)$ is the free in Θ algebra over X.

The set $Hom(W(X), H)$ of all homomorphisms from $W(X)$ to H is regarded as an affine space or, what is the same, a space of points.

Every point defined in such a way has the classical kernel $Ker(\mu)$ and, as we will see later, the logical kernel $LKer(\mu)$.

Along with free in Θ algebras $W(X)$ we consider also algebras of formulas $\Phi(X)$ which are also associated with the given Θ. We leave the precise definition of the algebra $\Phi(X)$ till Subsection 1.6. Right now we can note that $\Phi(X)$ is an extended boolean algebra which means that $\Phi(X)$ is a boolean algebra with the operations $\exists x$, $x \in X$ called existential quantifiers and with nullary operations of the form $w \equiv w'$, where $w, w' \in W(X)$, called equalities. There is a bunch of axioms regulating $\Phi(X)$.

We view all formulas $u \in \Phi(X)$ as equations. In particular, the formulas of the form $w \equiv w'$, where $w, w' \in W(X)$ are equations, since they are the elements of $\Phi(X)$.

Let a point $\mu : W(X) \to H$ be given. The logical kernel $LKer(\mu)$ consists of all formulas $u \in \Phi(X)$ valid on the point μ (see Definition 1.6). This is an ultrafilter in the algebra $\Phi(X)$.

1.3. **Route map.** We fix an infinite set of variables X^0 and the system Γ of all finite subsets X in X^0. Everywhere in the sequel we will ground on this infinite system Γ of finite sets instead of one infinite X^0. This is forced by the necessity to use logic in universal algebraic geometry. So, assume that all X in algebras $W(X)$ and $\Phi(X)$ belong to Γ.

Let us list in the order of appearance the main structures we are going to introduce.

- $-\Theta$, the variety of algebras Θ we started with,

- $-\Theta^0$, the category Θ^0 of all free algebra $W(X)$. Morphisms in Θ^0 are presented by homomorphisms $s : W(X) \to W(Y)$. Algebras $W(X) \times W(X)$ is the place, where the equations $w = w'$ live.

- $-\Theta(H)$, the category of affine spaces, see 1.4,

- $-Hal_\Theta(H)$, the category of algebras of the form $Hal_\Theta^X(H)$, see 1.4. Algebras $Hal_\Theta^X(H)$ are the very specific algebras, built on the base of affine spaces $Hom(W(X), H)$.

- $-Hal_\Theta$, the variety of multi-sorted Halmos algebras. All algebras $Hal_\Theta(H) = (Hal^X(H), \ X \in \Gamma)$, belong to this variety, see 1.5,

- $-\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$, the multi-sorted free in Hal_Θ over equalities algebra of formulas, see 1.6. Algebras $\Phi(X)$ is the place, where the formulas live.

- $-Hal_\Theta^0$, the category of algebras of formulas $\Phi(X)$, with morphisms $s_* : \Phi(X) \to \Phi(Y)$, see 1.6,

- $-Val_H = (Val_H^X, X \in \Gamma)$, the homomorphism calculating values of formulas from $\Phi(X)$ in algebras $Hal_\Theta^X(H)$, see 1.8.

In order to do that we shall define, first, the variety of multi-sorted Halmos algebras Hal_Θ. This variety Hal_Θ corresponds to the initial variety of algebras Θ. Then, we construct Hal_Θ^0 as a category, corresponding to a free multi-sorted algebra $\widetilde{\Phi}$ in Hal_Θ, built over a multi-sorted system of equalities.

We will build the morphisms s_* in Hal_Θ^0 in such a way, that the correspondence $s \to s_*$ determines a covariant functor from $\Theta^0 \to Hal_\Theta^0$. Thus, s_* has to preserve the boolean structure of the algebra

of formulas $\Phi(X)$ and be correlated with quantifiers and equalities. In particular,

$$s_*(w \equiv w') = (s(w) \equiv s(w')).$$

For other $u \in \Phi(X)$ the formula $s_* u = v \in \Phi(Y)$ is calculated in a more complicated way. Here, one can observe that formulas $v \in \Phi(Y)$ can contain also variables from $X \neq Y$. In fact, for $v \in \Phi(Y)$ there is no canonical form, which represent v through the variables.

We will start to fulfill this program from the next Section.

1.4. Algebras and categories $Hal_\Theta(H)$.

Algebras from the variety Hal_Θ will be the main structures in our setting. We approach to these algebras by introducing the category $Hal_\Theta(H)$. Let us start to do that.

For each algebra $H \in \Theta$ and every finite set $X \in \Gamma$ consider the algebra $Bool(W(X), H)$. This is the boolean power algebra of $Hom(W(X), H)$ with quantifiers $\exists x,\ x \in X$ and equalities.

Define, first, quantifiers $\exists x,\ x \in X$ on $Bool(W(X), H)$. Let A be set from $Bool(W(X), H)$. We set $\mu \in \exists x A$ if and only if there exists $\nu \in A$ such that $\mu(y) = \nu(y)$ for every $y \in X,\ y \neq x$. It can be checked that $\exists x$ defined in such a way is, indeed, an existential quantifier.

An equality $[w \equiv w']_H$ in $Bool(W(X), H)$ is defined by

$$\mu \in [w \equiv w']_H \Leftrightarrow (w, w') \in Ker(\mu).$$

For some reason we denote the obtained algebra by $Hal_\Theta^X(H)$. This is an example of the extended boolean algebra in the variety Θ. An algebra of formulas $\Phi(X)$ is the structure of this kind.

We will define the algebras $\Phi(X)$ in Subsection 1.6. In Subsection 1.6 we define the important homomorphisms

$$Val_H^X : \Phi(X) \to Hal_\Theta^X(H).$$

Define now the category $Hal_\Theta(H)$. Its objects are just defined algebras $Hal_\Theta^X(H)$, where H is given and $X \in \Gamma$. In order to define morphisms in $Hal_\Theta(H)$, consider first the category $\Theta(H)$ of affine spaces.

The objects of $\Theta(H)$ are affine spaces $Hom(W(X), H)$. Assign to each morphism $s : W(X) \to W(Y)$ the map $\tilde{s} : Hom(W(Y), H) \to Hom(W(Y), H)$ defined by the rule $\tilde{s}(\mu) = \mu s : W(X) \to H$, for $\mu : W(Y) \to H$. These $\tilde{s}$ are morphisms in $\Theta(H)$.

The correspondence $W(X) \to Hom(W(X), H)$ and $s \to \tilde{s}$ defines a contravariant functor $\Theta^0 \to \Theta(H)$ which determines duality.

Theorem 1.1. *The categories Θ^0 and $\Theta(H)$ are dually isomorphic under this functor if and only if $Var(H) = \Theta$.*

Morphisms s_*^H in $Hal_\Theta(H)$ are defined as follows. Every homomorphism $s : W(X) \to W(Y)$ gives rise to a Boolean homomorphism

$$s_*^H : Bool(W(X), H) \to Bool(W(Y), H),$$

defined by the rule: for each $A \subset Hom(W(X), H)$ the point μ belongs to s_*A if $\tilde{s}(\mu) = \mu s \in A$.

The defined category $Hal_\Theta(H)$ can be treated as a multi-sorted algebra

$$Hal_\Theta(H) = (Hal^X(H), \ X \in \Gamma),$$

with objects as domains and morphisms

$$s_*^H : Bool(W(X), H) \to Bool(W(Y), H),$$

as operations.

The algebra of formulas $\widetilde{\Phi} = Hal_\Theta^0 = (\Phi(X), \ X \in \Gamma)$ is defined in a similar way.

1.5. **Signature of algebras $Hal_\Theta(H)$.** Our next aim is to describe the signature of operations for the multi-sorted algebras $Hal_\Theta(H)$. This signature should be also multi-sorted.

Consider an arbitrary $W(X)$ in Θ, and take the signature

$$L_X = \{\vee, \wedge, \neg, \exists x, M_X\}, \ \text{for all } x \in X.$$

Here M_X is the set of all formulas $w \equiv w'$, $w, w' \in W(X)$ over the algebra $W(X)$. We treat these formulas as symbols of relations of equality over $W(X)$, that is there is a map $\equiv: W(X) \times W(X) \to M_X$ which satisfies axioms of an equational predicate on $W(X)$. These are the only symbols of relations in use. Symbols $w \equiv w'$ can be regarded also as symbols of nullary operations.

Signature L_X is the signature of the one-sorted extended boolean algebras. Now we define the signature L_Θ for the multi-sorted algebras $Hal_\Theta(H)$.

Along with the set of symbols of equalities M_X consider the set $S_{X,Y}$ of symbols of operations s_* of the type $\tau = (X; Y)$, where $X, Y \in \Gamma$. Symbols s_* are just symbols of operations but we keep in mind that each homomorphism $s : W(X) \to W(Y)$ induces the operation s_* in $Hal_\Theta(H)$ of the type $\tau = (X; Y)$.

By the same reason we assume that given $s : W(X) \to W(Y)$ and $s' : W(Y) \to W(Z)$, the axiom

$$(ss')_* = s_*s'_*$$

holds. Here the operation $(ss')_*$ has the type $\tau = (X; Z)$.

Define the signature

$$L_\Theta = \{L_X, S_{X,Y}; X, Y \in \Gamma\}$$

The signature L_Θ is multi-sorted. We consider the constructed multi-sorted algebras $Hal_\Theta(H)$ in this signature with the natural realization of all operations from L_Θ. We will take L_Θ also as the signature of an arbitrary algebra from the variety of multi-sorted algebras Hal_Θ.

Now we define algebras which belong to the variety Hal_Θ.

Definition 1.2. We call an algebra $\mathfrak{L} = (\mathfrak{L}_X, X \in \Gamma)$ in the signature L_Θ a Halmos algebra, if

(1) Every domain $\mathfrak{L}_X$ is an extended Boolean algebra in the signature L_X.

(2) Every mapping $s_* : \mathfrak{L}_X \to \mathfrak{L}_Y$ is a homomorphism of Boolean algebras. Let $s : W(X) \to W(Y)$, $s' : W(Y) \to W(Z)$, and let $u \in \mathfrak{L}_X$. Then $s'_*(s_*(u)) = (s's)_*(u)$.

(3) The identities controlling the interaction of s_* with quantifiers are as follows:

   (a) $s_{1*}\exists x a = s_{2*}\exists x a$, $a \in \mathfrak{L}(X)$, if $s_1(y) = s_2(y)$ for every $y \neq x$, $x, y \in X$.

   (b) $s_*\exists x a = \exists (s(x))(s_*a)$, $a \in \mathfrak{L}(X)$, if $s(x) = y$ and y is a variable which does not belong to the support of $s(x')$, for every $x' \in X$, and $x' \neq x$.
   This condition means that y does not participate in the shortest expression of the element $s(x') \in W(Y)$.

(4) The identities controlling the interaction of s_* with equalities are as follows:

   (a) $s_*(w \equiv w') = (s(w) \equiv s(w'))$.

   (b) $(s_w^x)_*a \wedge (w \equiv w') \leq (s_{w'}^x)_*a$, where $a \in \mathfrak{L}(X)$, and $s_w^x \in End(W(X))$ is defined by $s_w^x(x) = w$, and $s_w^x(x') = x'$, for $x' \neq x$.

One should not be upset with the looking complicated axioms from the items 3–4. First of all, we have already algebras $Hal_\Theta(H)$ as an example of Halmos algebras, So, one can verify in the very straightforward way that $Hal_\Theta(H)$ satisfy these axioms. Second, the purely logical explanations on the language of first order formulas of the similar axioms for the one-sorted polyadic algebras are contained in [5], see also [12].

Definition 1.3. The variety Hal_Θ consists of the multi-sorted algebras in the signature L_Θ subject to axioms from Definition 1.2.

The conditions specified in Definition 1.2 have the form of identities and they actually define a variety.

1.6. **Free algebras in Hal_Θ.** We construct the free in Hal_Θ algebra $\widetilde{\Phi}$ in an explicit way. Denote by $M = (M_X, X \in \Gamma)$ the multi-sorted set of equalities with the components M_X.

Let us build the absolutely free algebra of formulas in the signature L^0_Θ. Each formula in this algebra has two parameters: the length and the sort. Then we define formulas by induction.

Each equality $w \equiv w'$ is a formula of the length zero, and of the sort X if $w \equiv w' \in M_X$. Let u be a formula of the length n and the sort X. Then the formulas $\neg u$ and $\exists x u$ are the formulas of the same sort X and the length $(n + 1)$. Further, for the given $s : W(X) \to W(Y)$ we have the formula $s_* u$ with the length $(n + 1)$ and the sort Y. Let now u_1 and u_2 be formulas of the same sort X and the length n_1 and n_2 accordingly. Then the formulas $u_1 \vee u_2$ and $u_1 \wedge u_2$ have the length $(n_1 + n_2 + 1)$ and the sort X. In such a way, by induction, we define lengths and sorts of arbitrary formulas.

We construct a big set formulas $\mathfrak{L}^0$.

Let $\mathfrak{L}^0_X$ be the set of all formulas of the sort X. Each $\mathfrak{L}^0_X$ is an algebra in the signature L_X and we have the algebra

$$\mathfrak{L}^0 = (\mathfrak{L}^0_X, X \in \Gamma)$$

in the signature L_Θ. By construction, the algebra $\mathfrak{L}^0$ is the absolutely free algebra of formulas over equalities $M = (M_X, X \in \Gamma)$ concerned with the variety of algebras Θ. Its elements are considered as pure formulas.

Denote by $\tilde{\pi}$ the verbal congruence in $\mathfrak{L}^0$ generated by the identities of Halmos algebras from Definition 1.2.

Define the Halmos algebra of formulas as

$$\widetilde{\Phi} = \mathfrak{L}^0/\tilde{\pi}.$$

It can be written as $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$, where

$$\Phi(X) = \mathfrak{L}^0_X/\tilde{\pi}_X,$$

where each $\Phi(X)$ is an extended Boolean algebra of the sort X in the signature L_X.

The algebra $\widetilde{\Phi}$ is the free algebra in the variety Hal_Θ of all multi-sorted Halmos algebras associated with the variety of algebras Θ, with the set of free generators $M = (M_X, X \in \Gamma)$.

This approach to the free in Hal_Θ algebra through the factorization of the absolutely free algebra of formulas by the verbal congruence can be viewed as syntactic.

We could built $\widetilde{\Phi}$ also semantically. In each $\mathfrak{L}^0_X$ the formulas of the sort X are collected. Recall that for $s : W(X) \to W(Y)$ and $u \in \mathfrak{L}^0_X$

the formula $v = s_* u$ lies in $\mathfrak{L}_X^0$. All these formulas can be treated as pure formulas of a logic which possesses some axioms and rules of inference which correlate with the definition of the variety Hal_Θ.

One can show that if we factor out component-wisely the algebra $\mathfrak{L}^0$ by the many-sorted Lindenbaum-Tarski congruence, then we get the same algebra $\widetilde{\Phi}$. This observation provides a bridge between syntactical and semantical description of the free multi-sorted Halmos algebras.

Using the correspondence between multi-sorted algebras and categories we are able to define the category of algebras of formulas Hal_Θ^0. Its objects are algebras $\Phi(X)$, its morphisms have the form $s_* : \Phi(X) \to \Phi(Y)$, where $s : W(X) \to W(Y)$ is a morphism in Θ^0. According to Definition 1.2 the morphisms s_* preserve the boolean structure of $\Phi(X)$ and correlated with quantifiers and equalities. Hence the correspondence

$$W(X) \to \Phi(X) \quad \text{and} \quad s \to s_*$$

determines a covariant functor $\alpha : \Theta^0 \to Hal_\Theta^0$.

1.7. Variety Hal_Θ and algebras $Hal_\Theta(H)$. As we have seen, the algebras $Hal_\Theta(H)$ belong to Hal_Θ. Moreover,

Theorem 1.4. *Algebras $Hal_\Theta(H)$ generate the variety Hal_Θ.*

This means that we could define the algebra $\widetilde{\Phi}$ using algebras $Hal_\Theta(H)$. We have a unique homomorphism of the algebra $\mathfrak{L}$ to $Hal_\Theta(H)$. The kernel of this homomorphism is the system of identities of $Hal_\Theta(H)$. Intersection of all kernels through all $H \in \Theta$ and coincide with the verbal congruence with respect to the variety Hal_Θ

One can prove that these algebras are simple with respect to congruences and all simple algebras are exhausted by algebras $Hal_\Theta(H)$ and their subalgebras.

1.8. The Value homomorphism. The free algebra $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$ and an arbitrary algebra $Hal_\Theta(H) = (Hal_\Theta^X(H),\ X \in \Gamma)$ belong to the same variety Hal_Θ.

We define a homomorphism

$$Val_H : \widetilde{\Phi} \to Hal_\Theta(H),$$

which induces the homomorphism

$$Val_H^X : \Phi(X) \to Hal_\Theta^X(H)$$

of the one-sorted extended Boolean algebras, for every X in Γ.

Since equalities $M = (M_X, X \in \Gamma)$ freely generate the free multi-sorted Halmos algebra $\widetilde{\Phi}$, it is enough to assign an equality in $Hal_\Theta^X(H)$ to the corresponding equality $w \equiv w'$ in $\Phi(X)$.

Recall that we have defined equalities $[w \equiv w']_H$ in $Hal_\Theta^X(H)$ by

$$\mu \in [w \equiv w']_H \Leftrightarrow (w, w') \in Ker(\mu).$$

Hence, the element $[w \equiv w']_H$ in $Hal_\Theta^X(H)$ is assigned to the element $w \equiv w'$ in $\Phi(X)$. This correspondence gives rise to the homomorphism of multi-sorted Halmos algebras

$$Val_H : \widetilde{\Phi} \to Hal_\Theta(H).$$

Since $\widetilde{\Phi} = (\Phi(X), X \in \Gamma)$, where each component $\Phi(X)$ is an extended Boolean algebra, the homomorphism Val_H induces homomorphisms

$$Val_H^X : \Phi(X) \to Hal_\Theta^X(H),$$

of the one-sorted extended Boolean algebras. In particular,

$$Val_H^X(w \equiv w') = \{\mu \mid \mu(w) = \mu(w')\}.$$

Definition 1.5. A point $\mu : W(X) \to H$ satisfies the formula $u \in \Phi(X)$ if $Val_H^X(u)$ contains μ.

This definition has the same meaning as the standard model theoretic one. Now we are in a position to define formally the logical kernel of a point.

Definition 1.6. A formula $u \in \Phi(X)$ belongs to the logical kernel $LKer(\mu)$ of a point μ if and only if $\mu \in Val_H^X(u)$.

If $u \in \Phi(X)$ then $Val_H^X(u)$ is the set of points μ satisfying the formula u. This means that

$$u \in LKer(\mu) \Leftrightarrow \mu \in Val_H^X(u).$$

Since we consider each formula $u \in \Phi(X)$ as an "equation" and $Val_H^X(u)$ as a value of the formula u in the algebra $Bool(W(X), H)$, then $Val_H^X(u)$ is a set of points $\mu : W(X) \to H$ satisfying the "equation" u. We call $Val_H^X(u)$ *solutions of the equation u*. We also say that the formula u holds true in the algebra H at the point μ.

It can be verified that the logical kernel $LKer(\mu)$ is always a boolean ultrafilter of $\Phi(X)$.

Note that

$$Ker(\mu) = LKer(\mu) \cap M_X.$$

Now let $Th(H) = (Th^X(H), X \in \Gamma)$ be the multi-sorted representation of the elementary theory of H. We call its component $Th^X(H)$ the *X-theory of the algebra H*. Since $Ker(Val_H)$ is the set of formulas satisfied by all points of H, we have have:

$$Ker(Val_H) = Th(H),$$

$$Ker(Val_H^X) = Th^X(H).$$

This means, in particular, that the algebra $\Phi(X)$ can be represented, modulo elementary theory, in the more transparent algebra $Hal_\Theta^X(H)$. We can also present the X-theory of the algebra H as:

$$Th^X(H) = \bigcap_{\mu:W(X)\to H} LKer(\mu).$$

Definition 1.7. An algebra H is called saturated if for every finite X every ultrafilter T in $\Phi(X)$, which contains $Th^X(H)$ coincides with a $Lker(\mu)$ for some $\mu : W(X) \to H$.

This notion stimulates a lot of problems.

The key diagram which relates logic of different sorts in multi-sorted case is as follows:

$$
\begin{array}{ccc}
\Phi(X) & \xrightarrow{\ s_*\ } & \Phi(Y) \\
{\scriptstyle Val_H^X}\downarrow & & \downarrow{\scriptstyle Val_H^Y} \\
Hal_\Theta^X(H) & \xrightarrow{\ s_*\ } & Hal_\Theta^Y(H)
\end{array}
$$

Here the upper arrow represent the syntactical transitions in the category Hal_Θ, the lower level does the same with the respect to semantics in Hal_Θ, and the correlation is provided by the vertical value homomorphism.

With this diagram we finish exposition of the necessary ideas from algebraic logic and switch to a logically-geometric stuff. We defined formally the multi-sorted algebra of formulas $\tilde{\Phi}$ and its domains $\Phi(X)$, where $X \in \Gamma$. These domains could be informally treated as dynamic algebras of formulas, which means that formulas (elements) from a particular $\Phi(X)$ interact with formulas from other $\Phi(Y)$.

Now we have prepared all necessary information for the next Part.

2. **Some results and problems**

2.1. **Types and isotypeness.** In model theory for each set X, $X = \{x_1, \ldots, x_n\}$ the notion of X-type (n-type) is defined (see [7]). Given an algebra H with the elementary theory $Th(H)$, a set P of formulas u, such that all free variables in u belongs to $X = \{x_1, \ldots, x_n\}$ is an X-type (n-type), if $P \cup Th(H)$ is satisfiable. Denote by $T_p^H(\mu)$ the X-type of the point $\mu : W(X) \to H$, i.e., the set of all formulas u valid on μ. The types of the form $T_p^H(\mu)$ will be called *MT-types or model-theoretic types* of points.

We consider also *LG-types or logically-geometric types of the points* $\mu : W(X) \to H$. According to Definition 6.2 from [17] an *LG*-type of

a point μ is the logical kernel $LKer(\mu)$ of the point μ in the algebra $\Phi(X)$. We will return to the geometric nature of this definition later. The following result connects MT-types and LG-types, and plays a key role in all considerations:

Theorem 2.1 ([22]). *Let the points* $\mu : W(X) \to H_1$ *and* $\nu : W(X) \to H_2$ *be given. The equality*

$$T_p^{H_1}(\mu) = T_p^{H_2}(\nu)$$

holds if and only if we have

$$LKer(\mu) = LKer(\nu).$$

Now we will describe the idea of isotypeness of algebras.

Definition 2.2. Two algebras H_1 and H_2 are called isotypic if for every X and every point $\mu : W(X) \to H_1$ there exists a point $\nu : W(X) \to H_2$ such that the types of μ and ν coincide, and for for every point $\nu : W(X) \to H_2$ there exists a point $\mu : W(X) \to H_1$ such that the types of μ and ν coincide.

• – In view of Theorem 2.1 two algebras H_1 and H_2 are isotypic with respect to MT-types (MT-isotypic) if and only if they are isotypic with respect to LG-types (LG-isotypic). Thus, in Definition 2.2 one can equally rely on MT-types and LG-types.

• – Since in the definition of isotypeness one can grounds on coincidence of logical kernels of the points, then this definition has a geometric nature and extends the notion of geometrically equivalent algebras (see next Section 2.3 and especially Theorem 2.13 for details). One can say that Theorem 2.1 and Theorem 2.13 visualize the bridge between geometrical and logical ideas.

• – According to Definition 2.2 isotypeness of algebras implies their elementary equivalence. So, this notion is more strong than the notion of elementary equivalence and expresses a logical property of algebras which should in many cases be closed to isomorphism.

Indeed, it is easy to see that if algebras H_1 and H_2 are isotypic, then they are locally isomorphic. This means that every finitely generated subalgebra in H_1 is isomorphic to a finitely generated subalgebra in H_2 and vice versa. Of course, locally isomorphic groups are not necessarily isomorphic and isotypic. For example every two free groups F_m and F_n are locally isomorphic, but not isomorphic provided $m \neq n$. As will see F_m and F_n are not also isotypic. This follows from the result of C. Perin and R. Sklinos [10] on logical homogenity (see Section 2.4) of a free group.

Moreover, it easy to see that if $H_1 = F_n$ is a finitely generated free group and H_2 is a finitely generated group, then their isotypeness implies isomorphism. This follows from the local isomorphism of H_1 and H_2, which means that H_2 can be viewed as a subgroup of H_1. Since every subgroup of F_n is free, it remains to use the above mentioned result on isotypeness of free finitely generated groups.

We come up with the following conjecture:

Conjecture 2.3. *Let F_n be a free group of the rank $n > 1$ and H be a group. If F_n and H are isotypic then they are isomorphic.*

Recently, R. Sklinos [21] gave a positive answer to this problem. His proof is also based on the logical homogenity of a free group in the form of the following theorem of Pillay:

Theorem 2.4 (Pillay). *Let F_n be the free group with free generators $e_1, \ldots, e_n$. Consider the points $\mu : W(X) \to F_n$ and $\nu : W(X) \to F_n$ defined by $\mu(x_i) = e_i$ and $\nu(x_i) = a_i$, respectively, where $i = 1, \ldots n$, and a_i are arbitrary elements in F_n. Suppose that*

$$T_p^H(\mu) = T_p^H(\nu).$$

Then $a_1, \ldots, a_n$ are the free generators of F_n.

We see that all groups isotypic to a free finitely generated group F_n should be isomorphic to F_n. The next problems are related to the general case:

Problem 2.5. Let the groups H_1 and H_2 be isotypic and H_1 be finitely generated. Is it true that H_2 is finitely generated?

Problem 2.6. Let H_1 and H_2 be two finitely generated isotypic groups. Are they isomorphic?

Since all the theory grounds on an arbitrary variety of algebras Problems 2.6 and 2.5 which are formulated for the variety of all groups make sense for an arbitrary variety of algebras Θ. Their solution heavily depends on the choice of Θ. So, let H_1 and H_2 be two algebras from a variety Θ.

Problem 2.7. Let the algebras H_1 and H_2 be isotypic and H_1 be finitely generated. Is it true that H_2 is finitely generated?

Problem 2.8. Let H_1 and H_2 be two finitely generated isotypic algebras. Are they isomorphic?

Let us point out one more closely related question

Problem 2.9. Let H_1 and H_2 be two finitely generated isotypic groups. What can be said about isotypeness of the groups algebras KH_1 and KH_2 where K is a field.

As we have mentioned above a local isomorphism does not imply isotypeness. Here is an example when local isomorphism of algebras implies isotipicity:

Example 2.10. Any two infinite dimensional vector spaces H_1 and H_2 are locally isomorphic. It can checked that their local isomorphism implies isotypeness of H_1 and H_2. Take now two non-isomorphic infinitely dimensional vector spaces. Then, in view of above they provide an example of isotypic, locally isomorphic but not isomorphic algebras.

2.2. **Algebraic sets.** Recall some basic notions from equational algebraic geometry in the given variety Θ. Fix a finite set $X = \{x_1, \ldots, x_n\}$ and let T be a system of equations of the form $w \equiv w'$, $w, w' \in W(X)$.

Define
$$T'_H = A = \{\mu : W(X) \to H \mid$$
$$T \subset Ker(\mu)\}$$

Subsets A in $Hom(W(X), H)$ of the form $A = T'_H$ are called *algebraic sets*.

Let A be an arbitrary set of points $\mu : W(X) \to H$ in $Hom(W(X), H)$. Define
$$A'_H = T = \{(w \equiv w') \mid$$
$$(w, w') \in \bigcap_{\mu \in A} Ker(\mu)\}.$$

The congruences T of the form $T = A'_H$ are called H-closed congruences in $W(X)$. We obtain a Galois correspondence between different T and A. If A and T are arbitrary sets of points and sets of equations, respectively, then A''_H and T''_H are the corresponding Galois closures.

Define now an important notion:

Definition 2.11. Algebras H_1 and H_2 are called geometrically equivalent (or AG-equivalent) if for every finite X and every T in $W(X)$ we have
$$T''_{H_1} = T''_{H_2}.$$

A criterion for the geometrical equivalence is a follows :
Consider infinitary quasiidentities of the form $(*)$

$$w_1 \equiv w'_1 \wedge \ldots \wedge w_n \equiv w'_n \wedge \ldots$$

$$\Longrightarrow w_0 \equiv w_0'.$$

Then, H_1 and H_2 are AG-equivalent if and only if every quasiidentity of the form $(*)$ which holds in H_1 holds also in H_2 and vice versa. In particular, if H_1 and H_2 are AG-equivalent, then they have the same finitary quasiidentities. The converse is not true [4], [8].

One can define in a very natural way the category of algebraic sets over the given algebra H in the variety Θ. Denote it by $K_\Theta(H)$.

It is easy to prove that if algebras H_1 and H_2 are AG-equivalent, then the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$ are isomorphic. In the paper [14] there are necessary and sufficient conditions for specific algebras H_1 and H_2 that provide isomorphism of the categories $K_\Theta(H_1)$ and $K_\Theta(H_2)$.

2.3. **Elementary (definable) sets.** Now we switch to logical geometry. The main correspondences in logical geometry repeat that of algebraic geometry with replacing kernels of points $Ker(\mu)$ with their logical kernels $LKer(\mu)$.

If T is set of formulas in the algebra $\Phi(X)$ then define:

$$T_H^L = A = \{\mu : W(X) \to H \mid$$
$$T \subset LKer(\mu)\},$$

Subsets A in $Hom(W(X), H)$ of the form $A = T_H^L$ are called *elementary or definable sets.*

Define

$$A_H^L = T = \bigcap_{\mu \in A} LKer(\mu)$$

The filters T of the form $T = A_H^L$ are called H-closed filters in $\Phi(X)$. We obtain a Galois correspondence between T and A.

We have also that a formula $u \in \Phi(X)$ belongs to $T = A_H^L$ if and only if $A \subset Val_H^X(u)$.

Definition 2.12. Algebras H_1 and H_2 are called logically equivalent (or LG-equivalent) if for every finite X and every $T \subset \Phi(X)$ we have

$$T_{H_1}^{LL} = T_{H_2}^{LL}.$$

Using infinitary formulas of the form $(*)$:

$$\bigwedge_{u \in T} u \Longrightarrow v, \quad u, v \in \Phi(X)$$

we can formulate the criterion:

Algebras H_1 and H_2 are LG-equivalent if and only if every formula of the form $(*)$ which holds in H_1 holds also in H_2 and vice versa.

The main point is the following theorem which connects the model theoretic notion of isotypic algebras with the geometric notion of LG-equivalent algebras:

Theorem 2.13. *Algebras H_1 and H_2 are isotypic if and only if they are LG-equivalent.*

Consider the category of elementary sets over the given algebra H in the variety Θ. Denote it by $LK_\Theta(H)$.

Theorem 2.14. *If algebras H_1 and H_2 are isotypic, then the categories $LK_\Theta(H_1)$ and $LK_\Theta(H_2)$ are isomorphic.*

In the situation of logical geometry we do not have yet necessary and sufficient conditions on algebras H_1 and H_2 that provide isomorphism of the categories $LK_\Theta(H_1)$ and $LK_\Theta(H_2)$.

One can prove that every elementary set A in $Hom(W(X), H)$ is invariant under the action of the group $Aut(H)$ in this affine space.

2.4. **Logically regular varieties.** Now,

Definition 2.15. Algebra $H \in \Theta$ is called logically separable (in Θ) if every $H' \in \Theta$ which is not isomorphic to H is not isotypic to H.

This means that algebra $H \in \Theta$ is logically separable (in Θ) if every $H' \in \Theta$ isotypic to H is isomorphic to H. Definition 2.15 says that a logically separable algebra H can be distinguished by means of the logic of types. We are looking for the cases, when a free in Θ algebra $W(X)$, $X \in \Gamma$ is separable in Θ.

Definition 2.16. A variety Θ is called LG-regular (logically regular) if every free in Θ algebra $W(X)$, $X \in \Gamma$ is logically separable.

It is checked that the varieties of semigroups, of inverse semigroups are LG-regular (see [22]), the variety of abelian groups is LG-regular (see [22] and [20]).

More recently it was established that:

- – variety of all groups is LG-regular [21],

- – variety of all nilpotent of class c groups is LG-regular [22].

Problem 2.17. Is it true that the following varieties are LG-regular:

- – Variety of solvable groups of the derived length d.

- – Variety of commutative associative algebras over a field.

- – Variety of associative algebras over a field.

- – Variety of Lie algebras.

- – Variety of class c nilpotent associative algebras.

- – Variety of class c nilpotent Lie algebras.

We shall specify also the following:

Problem 2.18. Find a variety Θ which is not LG-regular. Construct a non-regular variety of groups.

2.5. Logically homogeneous algebras.

Definition 2.19. An algebra $H \in \Theta$ is called logically homogeneous if for every two points $\mu : W(X) \to H$ and $\nu : W(X) \to H$ the coincidence of their types implies that there exists an automorphism σ of H such that $\sigma(\mu) = \nu$.

It can be seen that if an algebra H is logically homogeneous, then for every X every orbit of the action of $Aut(H)$ in $Hom(W(X), H$ is an elementary set for some T in $\Phi(X)$.
Take an arbitrary point $\mu : W(X) \to H$, and consider $T = LKer(\mu)$ in $\Phi(X)$. Then the elementary set T_H^L defined by the ultrafilter T is an orbit of $Aut(H)$ containing the point μ, provided H is logically homogeneous.

Definition 2.20. We call a variety Θ logically perfect if every free in Θ algebra $W(X)$ is logically homogeneous.

Problem 2.21. What are the varieties Θ such that an arbitrary free in Θ algebra $W(X)$, $X \in \Gamma$ is logically homogeneous.

Problem 2.21 has positive solution for the variety of all groups [10], for the varieties of abelian groups and nilpotent of class c groups ([22], for abelian groups see also [20]). It is also proved that the torsion free hyperbolic groups are logically homogeneous [9].

Problem 2.22. Is it true that the variety of solvable groups of the derived length $d > 1$ is logically perfect?

In particular,

Problem 2.23. Is it true that the variety of metabelian groups is logically perfect?

Note, that up to now we don't have examples of non-logically homogeneous varieties of groups Θ.

Definition 2.24. Algebra H is called algebraically homogeneous if for two points $\mu : W(X) \to H$, and $\nu : W(X) \to H$ the equality $Ker(\mu) = Ker(\nu)$ implies that there exists an automorphism σ of H such that $\sigma(\mu) = \nu$.

It is clear that the algebraic homogenity means that every isomorphism $\alpha : A \to B$ of their finitely generated subalgebras A and B is realized by some $\tau \in Aut(H)$. By definition, algebraic homogenity implies logical homogenity. Converse statement is not true. For example, the free abelian group is logically homogeneous, but no algebraically homogeneous.

Algebraically homogeneous solvable groups and finite groups are classified, see [1], and [2], respectively. These groups are very closed to the QE-groups, i.e., the groups whose elementary theory admits quantifier elimination. The similar theory is elaborated for algebraically homogeneous rings (see [3] and references therein).

2.6. **Logical noetherianity.** Consider now logically noetherian algebras.

Definition 2.25. An algebra $H \in \Theta$ is called logically noetherian (LG-noetherian) if for every $X \in \Gamma$ and every $T \subset \Phi(X)$ there exists a finite subset $T_0 \subset T$ such that

$$T_H^L = T_{0H}^L.$$

In concern with this definition an algebra $H \in \Theta$ is called automorphically finitary if there is only a finite number of $Aut(H)$ -orbits in $Hom(W(X), H)$ for every X. If H is automorphically finitary, then H is LG-noetherian.

Problem 2.26. 1. Describe automorphically finitary abelian groups.
2. Consider non-commutative automorphically finitary groups.

If for the algebra H there are only a finite number of $Aut(H)$-orbits in $Hom(W(X), H)$ for every X, then there are only finite number of realizable LG-types in $\Phi(X)$.

2.7. **Addendum.** This paper is devoted to the theory, which gave rise to a system of new notions. It arises in a natural way and stimulates a lot of new problems in algebra and universal algebraic geometry which are concerned with logic and algebraic logic. Recent results of G.Zhitomirskii and R.Sklinos are of this kind. We also look at new papers of Z.Sela [19] , and O.Kharlampovich and A.Myasnokov [6] through the prism of the described theory.

Note the following general

Theorem 2.27. *If in the variety of algebras* Θ *is logically perfect, then* Θ *is LG-regular.*

We know that the varieties of all groups, the variety of nilpotent of class c groups, the variety of abelian groups are *LG*-perfect. Hence, these varieties are *LG*-regular. This makes the problem if the varieties of solvable groups of the derived length d, of metabelian groups and of commutative associative rings with unity are perfect, especially interesting. We have no examples of varieties which are not logically perfect and are not logically regular. So, it is quite important to check the varieties specified above. More precisely, is it possible in these cases that the group of automorphisms does not act transitively on Galois closures of points?

We will finish with the problems related to elementary equivalence of algebras. These problems, however, appeared as a result of studies around isotypeness of algebras.

Problem 2.28. Find a variety of groups Θ, which is distinct from the variety of all groups, such that all free in Θ finitely generated algebras are elementary equivalent, but if they are isotypic, then they are isomorphic. In particular,

 • – Is the Burnside variety B_n of groups satisfying the identity $x^n = 1$, of this kind?
 • – Is the Engel variety E_n of groups satisfying the identity

$$[x, y] \ldots, y] = 1,$$

where the commutator is taken n-times, of this kind?

Problem 2.29. Find two elementary equivalent groups H_1 and H_2 such that their group algebras are elementary equivalent, but not isotypic.

References

[1] G. Cherlin, U. Felgnet, Homogeneus solvable groups, *J. London Math. Soc.*,**44**, (1991) 102–120.
[2] G. Cherlin, U. Felgnet, The classification of finite homogeneous groups, *J. London Math. Society*, **62**, (2000), 784–794.
[3] G. Cherlin, D. Saracino and C. Wood, On homogeneous nilpotent groups and rings, *Proceedings American Math. Society*, **119** (1993), 1289–1306.
[4] R. Gobel, S. Shelah, Radicals and Plotkin's problem concerning geometrically equivalent groups. *Proc. Amer. Math.Soc.*, **130** (2002), 673–674.
[5] P.R. Halmos, *Algebraic logic*, New York, 1969.
[6] O. Kharlampovich, A. Myasnikov, Definable subsets in a hyperbolyc group, preprint arXiv: 1111.0577v2 [math.GR].

[7] D. Marker, *Model Theory: An Introduction* (Springer Verlag, 2002).

[8] A. Myasnikov, V. Remeslennikov, Algebraic geometry over groups II, Logical foundations, *J. of Algebra* **234** (2000) 225–276.

[9] A. Ould Houcine, Homogenity and prime models in torsion-free hyperbolic groups, *Confluentes Mathematici* **3** (2011) 121–155.

[10] C. Perin, R. Sklinos, Homogenety in the free group, *Duke Mathematical Journal*, to appear. Preprint arXiv: 1003.4095v1 [math.GM].

[11] A. Pillay, On genericity and weight in the free group, *Proc. Amer. Math. Soc.* **137** (2009), 39113917.

[12] B. Plotkin *Universal algebra,algebraic logic and databases.* Kruwer Acad. Publ., (1994).

[13] B. Plotkin *Algebra, categories, and databases,* Handbook of Algebra, **v.2**, Elsevier (2000), 81–148. (1994).

[14] B. Plotkin, *Algebras with the same algebraic geometry*, Proceedings of the International Conference on Mathematical Logic, Algebra and Set Theory, dedicated to 100 anniversary of P.S.Novikov, Proceedings of the Steklov Institute of Mathematics, MIAN, **v.242**, (2003), 176 – 207. Arxiv: math.GM/0210194.

[15] B. Plotkin, *Algebraic geometry in First Order Logic*, Sovremennaja Matematika and Applications **22** (2004), 16–62. Journal of Math. Sciences, **137**, n.5, (2006), 5049– 5097.

[16] B. Plotkin, T. Plotkin, Geometrical aspects of databases and knowledge bases, *Algebra Universalis*, (2012), to appear.

[17] B. Plotkin, E. Aladova, E. Plotkin, Algebraic logic and logically-geometric types in varieties of algebras, *Journal of Algebra and its Applications*, (2012), to appear.

[18] B. Plotkin, E. Aladova, E. Plotkin, Algebraic logic and logical geometry in arbitrary varieties of algebras, in Progress.

[19] Z. Sela, Diophantine geometry over groups X: The elementary theory of free products of groups, preprint arXiv: 1012.0044v1 [math.GR]

[20] R. Sklinos, Private correspondence.

[21] R. Sklinos, Unpublished.

[22] G. Zhitomirski, On logically-geometric types of algebras, preprint arXiv: 1202.5417v1 [math.LO].

Boris Plotkin: Institute of Mathematics, Hebrew University, 91904, Jerusalem, ISRAEL

E-mail address: `plotkinacs.biu.ac.il`